Rephasing Invariant Parametrization of Flavor Mixing Matrices

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The three-flavor mixing matrix can be parameterized by the rephasing invariants $\Gamma_{ijk} = V_{1i}V_{2j}V_{3k}$. This formulation brings out the inherent symmetry of the problem and has some appealing features. Examples illustrating the parametrization and applications to quark mixing are presented.

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I.   INTRODUCTION

It is well-known that the flavor mixing matrices of quarks (V_{CKM}) and neutrinos (V_ν) can be multiplied by phase matrices (rephasing) without changing their physical contents. Thus, amongst the full set of parameters of these matrices (nine for $U(3)$ and eight for $SU(3)$), only four are physical. The choice of these physical parameters are by no means unique. In fact, a number of them are in common usage. One may choose three mixing angles and a phase, as in the original Cabibbo-Kobayashi-Maskawa (CKM) parametrization [1], or the "standard parametrization" in the particle data book [2], or other similar schemes [3]. For V_{CKM}, a very convenient choice turns out to be the Wolfenstein parametrization [4], which exhibits the magnitude of the matrix elements clearly, even though the rephasing angles are fixed in a specific way. One could also use the absolute values $|V_{ij}|$ [5], which are manifestly rephasing invariant, although it is not clear which four of these nine should be singled out. Similarly, another choice is to use four of the nine rephasing invariants $V_{ik}V_{j\ell}V_{i\ell}^*V_{jk}^*$ [6].

In this paper we suggest yet another parametrization based on rephasing invariants. Without loss of generality, we consider only mixing matrices with $\det V = 1$. There are then six rephasing invariants $\Gamma_{ijk} = V_{1i}V_{2j}V_{3k}$, $(i, j, k) = $ permutation of $(1, 2, 3)$. They are shown to satisfy two simple constraints, leaving us with four independent ones. These Γ's are found to be closely related to the other rephasing invariants, $|V_{ij}|^2$ and $V_{ik}V_{j\ell}V_{i\ell}^*V_{jk}^*$. However, they retain a lot of the symmetry inherent in the problem and their construction is equally valid for V_{CKM} as for V_ν. These features seem to be rather appealing, theoretically. We hope that their use can help to further our understanding of the flavor mixing problem.

In Sec.II, we define the rephasing invariants Γ_{ijk} and exhibit the two constraints which reduce the number of independent parameters to four. Sec.III is devoted to a description of their detailed properties. Applications to the quark mixing matrix will be presented in Sec.IV. Finally, some concluding remarks are offered in Sec.V.

II.   REPHASING INVARIANT PARAMETRIZATION

As we mentioned in the Introduction, there are several known parametrizations of the flavor mixing matrix. A common drawback of these schemes is the lack of uniqueness. For instance, there are many ways to choose the mixing angles because of non-commmutativity [3]. Similarly, it is not clear which four of the nine quantities, $|V_{ij}|^2$ or $V_{ik}V_{j\ell}V_{i\ell}^*V_{jk}^*$, should be favored. Despite arguments preferring one choice over another, it seems fair to say that a general criterion for a "best" set is still absent. We will now introduce yet another parametrization, which, in our opinion, alleviates the above problem to a large extent.

We begin by considering, without loss of generality, only mixing matrices which satisfy

$$\det V = +1, \tag{1}$$

i.e., only $SU(3)$, but not $U(3)$, matrices are used. Note that, while the "standard parametrization" satisfies Eq.(1), the original KM matrix does not. Eq.(1) implies that, in the rephasing transformation, $V \to V' = PVP'$, we can impose on the diagonal phase matrices the conditions, $\det P = \det P' = 1$. It follows immediately that we can construct a set of six rephasing invariants [7],

$$\Gamma_{ijk} = V_{1i}V_{2j}V_{3k}, \tag{2}$$

where $(i, j, k) =$ permutations of (1, 2, 3). These Γ's satisfy the constraints $(\det V = 1)$,

$$\sum(\pm)\Gamma_{ijk} = 1, \tag{3}$$

where the +(-) sign applies when (i, j, k) is an even (odd) permutation of (1, 2, 3). Let us define a matrix v, satisfying

$$\sum V_{ij}v_{ik} = \sum V_{ji}v_{ki} = \delta_{jk}, \tag{4}$$

i.e., v_{ij} is the cofactor of V_{ij}. Then, from $VV^\dagger = 1 = \det V$,

$$V_{ij}^* = v_{ij}. \tag{5}$$

E.g., $V_{11}^* = V_{22}V_{33} - V_{23}V_{32}$, $V_{12}^* = -(V_{21}V_{33} - V_{23}V_{31})$, etc. Using these equalities, we can relate Γ_{ijk} to $|V_{\ell m}|^2$. For instance,

$$|V_{12}|^2 = V_{12}V_{12}^*.$$
$$= V_{12}(-V_{21}V_{33} + V_{23}V_{31}) \tag{6}$$
$$= \Gamma_{231} - \Gamma_{213}$$

Similarly, all the $|V_{ij}|^2$ are equal to the differences of the Γ's. Thus, they must all have the same imaginary part,

$$\Gamma_{ijk} = R_{ijk} - iJ, \tag{7}$$

where R_{ijk} is real and J can be identified with the familiar CP-violation measure as follows. We define [6]

$$\Pi_{\alpha i} = V_{\beta j}V_{\gamma k}V_{\beta k}^* V_{\gamma j}^*, \tag{8}$$

where (α, β, γ) and (i, j, k) are cyclic permutations of (1, 2, 3), with

$$Im\Pi_{\alpha i} = J. \tag{9}$$

Using Eq.(5), we have, for instance,

$$\Pi_{11} = V_{22}V_{33}V_{23}^* V_{32}^*$$
$$= (V_{11}^* + V_{23}V_{32})(V_{23}^* V_{32}^*) \tag{10}$$
$$= \Gamma_{132}^* + |V_{23}|^2|V_{32}|^2,$$

establishing $Im\Gamma_{132} = -J$. At the same time, we may eliminate $V_{23}^* V_{32}^*$ in Π_{11} and find

$$\Pi_{11} = -\Gamma_{123} + |V_{22}|^2|V_{33}|^2, \tag{11}$$

i.e., the sum $(R_{123} + R_{132})$ is simply related to a combination of products of the $|V_{ij}|^2$'s.

The above results, with all possible choices of indices, can be collected in a compact form. Let us define the matrix.

$$W = \begin{pmatrix} |V_{11}|^2 & |V_{12}|^2 & |V_{13}|^2 \\ |V_{21}|^2 & |V_{22}|^2 & |V_{23}|^2 \\ |V_{31}|^2 & |V_{32}|^2 & |V_{33}|^2 \end{pmatrix} \tag{12}$$

together with the matrix w ($w_{ij} = $ cofactor of W_{ij}) defined by

$$\sum_i W_{ij}w_{ik} = \sum_i W_{ji}w_{ki} = (\det W)\delta_{jk}. \tag{13}$$

Thus, for instance, $w_{12} = $ - $(|V_{21}|^2|V_{33}|^2 - |V_{23}|^2|V_{31}|^2)$. We further separate the even and odd permutation R_{ijk}'s by defining

$$(x_1, x_2, x_3) = (R_{123}, R_{231}, R_{312}); \tag{14}$$
$$(y_1, y_2, y_3) = (R_{132}, R_{213}, R_{321}).$$

The relation $\det V = 1$ now reads

$$(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1. \tag{15}$$

The relations between $|V_{ij}|^2$ and R_{ijk} are summarized in

$$W = \begin{pmatrix} x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\ x_3 - y_2 & x_1 - y_3 & x_2 - y_1 \\ x_2 - y_3 & x_3 - y_1 & x_1 - y_2 \end{pmatrix}, \tag{16}$$

$$w = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ x_3 + y_2 & x_1 + y_3 & x_2 + y_1 \\ x_2 + y_3 & x_3 + y_1 & x_1 + y_2 \end{pmatrix}. \tag{17}$$

These equations enable one to switch between the two sets of parameters, (x_i, y_j) and $|V_{\ell m}|^2$.

We now turn to the relations between J and R_{ijk}

$$|\Gamma_{ijk}|^2 = |V_{1i}|^2|V_{2j}|^2|V_{3k}|^2 = |R_{ijk}|^2 + J^2. \tag{18}$$

Using Eqs.(14, 16), there result six such equations with $i = 1, 2, 3$,

$$(x_i - y_1)(x_i - y_2)(x_i - y_3) - x_i^2 = J^2, \tag{19}$$

$$(x_1 - y_i)(x_2 - y_i)(x_3 - y_i) - y_i^2 = J^2. \tag{20}$$

These are consistency conditions which, more interestingly, may be regarded as cubic equations whose solutions are the x's and y's. Thus, x_i are the three solutions of

$$\xi^3 - (1 + \sum y_j)\xi^2 + (y_1 y_2 + y_2 y_3 + y_3 y_1)\xi - (J^2 + y_1 y_2 y_3) = 0. \tag{21}$$

Likewise, y_i are those of

$$\eta^3 + (1 - \sum x_j)\eta^2 + (x_1 x_2 + x_2 x_3 + x_3 x_1)\eta + (J^2 - x_1 x_2 x_3) = 0. \tag{22}$$

It follows that

$$(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1, \tag{23}$$

$$x_1 x_2 + x_2 x_3 + x_3 x_1 = y_1 y_2 + y_2 y_3 + y_3 y_1. \tag{24}$$

In addition,

$$J^2 = x_1 x_2 x_3 - y_1 y_2 y_3. \tag{25}$$

Note that Eqs.(24, 25) also follow from Eq.(15) and the identity $\Gamma_{123}\Gamma_{231}\Gamma_{312} = \Gamma_{132}\Gamma_{213}\Gamma_{321}$. Thus, a rephasing invariant parametrization of V, with $\det V = 1$, consists of the set (x_i, y_j) subject to the two constraints in Eqs.(23, 24). Further, the CP-violation measure is given by the very appealing expression in Eq.(25). One may obtain four independent parameters out of the set (x_i, y_j) by eliminating any two of them through Eqs.(23, 24). However, it is clear that doing so would lose much of the inherent symmetry of the problem.

At this juncture it is instructive to compare our results with those of two flavor mixings. For $SU(2)$,

$$V = e^{i\delta\sigma_3} e^{i\theta\sigma_2} e^{i\delta'\sigma_3} \tag{26}$$

$$= \begin{pmatrix} g & h \\ -h^* & g^* \end{pmatrix}, \tag{27}$$

$(|g|^2 + |h|^2 = 1)$, and we may parameterize V either by θ or by one of the $|V_{ij}|$'s, say, $|V_{11}|^2 = |g|^2 = \cos^2 \theta$. However, one may equally well have used the (real) rephasing invariant parameters defined by

$$x = \Gamma_{12} = V_{11}V_{22} = \cos^2 \theta, \tag{28}$$

$$y = \Gamma_{21} = V_{12}V_{21} = -\sin^2 \theta, \tag{29}$$

$$x - y = \det V = 1. \tag{30}$$

While the generalizations to three flavors of the first two parameterizations are well-known, that of the third leads to the set (x_i, y_j) which was studied above.

III. PROPERTIES OF THE PARAMETRIZATION

We now turn to some detailed properties of the parameters (x_i, y_j). Let's start by establishing the range of their values. First, from Eq.(16), we have

$$(y_1, y_2, y_3) \leq (x_1, x_2, x_3). \tag{31}$$

Next, with $W_{ij} \leq 1$ and $|w_{ij}| \leq 1$, and relations Eqs.(16, 17) such as $x_1 = \frac{1}{2}(W_{11} + w_{11})$, etc., we readily find

$$-1 \leq (x_i, y_j) \leq 1. \tag{32}$$

Consistency of the constraint, Eq.(24) with Eqs.(31, 32) can now be used to establish that at most one x_i can be negative (and, similarly, only one y_j can be positive) and that

$$x_1 x_2 + x_2 x_3 + x_3 x_1 \geq 0. \tag{33}$$

Finally, it is not hard to deduce that

$$0 \leq (x_1 + x_2 + x_3) \leq 1, \tag{34}$$

$$-1 \leq (y_1 + y_2 + y_3) \leq 0. \tag{35}$$

To summarize, the parameters (x_i, y_j) are distributed in the interval [-1, 1], with $x_i \geq y_j$, for all (i, j). Also, $0 \leq \sum x_i \leq 1$, with at most one negative x_i, while $-1 \leq \sum y_j \leq 0$, with at most one positive y_j.

Turning to the matrices W and w, with $Ww^T = (\det W)I$, we find

$$\det W = (x_1^2 + x_2^2 + x_3^2) - (y_1^2 + y_2^2 + y_3^2), \tag{36}$$

which, by Eqs.(23, 24), reduces to

$$\det W = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3). \tag{37}$$

It follows, by Eqs.(34, 35), that

$$-1 \leq \det W \leq 1. \tag{38}$$

It is interesting to note that, while the elements of any row or column in W sum up to unity, the corresponding sum in w is equal to $\det W$, $\sum_i w_{iI} = \sum_i w_{Ii} = \det W$.

The properties discussed above also suggest an interesting relation between three flavor and two flavor mixing, as contained in the correspondence: $(\sum x_i) \rightarrow x$ and $(\sum y_j) \rightarrow y$. For two flavors, $\det W$ $(= x + y = \cos 2\theta)$ can be regarded as a measure of the deviation of the mixing from identity, with $\det W = 0$ at maximal mixing. It seems reasonable to use $\det W$ as a measure of the total amount of mixing for three flavors, with a necessary condition for maximal mixing again being $\det W = 0$.

We now consider two concrete examples which should help to illuminate the nature of the (x_i, y_j) parametrization.

A) $x_1 = 1, x_2 = x_3 = y_1 = \ldots = 0$.

This solution of course corresponds to the identity matrix, $V = I$, with $\det W = +1$. Cyclic permutations of the states generate equivalent solutions, with some $x_i = 1$ while all other parameters vanish. An exchange of the states switches the roles of x with y, resulting in solutions such as $y_1 = -1, x_1 = \ldots = 0$, with $\det W = -1$.

The physical quark mixing matrix, V_{CKM}, is very close to the identity matrix, with $x_1 \simeq 1$ and all other parameters $\simeq 0$. We will give a detailed description of V_{CKM} in the next section.

B) $x_1 = x_2 = x_3 = 1/6, y_1 = y_2 = y_3 = -1/6$.

The solution exhibits maximal symmetry for three flavor mixing, with

$$W = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \tag{39}$$

$\det W = 0$ and $w_{ij} = 0$. It is the three flavor generalization of the maximal mixing solution for the flavors, where the 2×2 W matrix is given by

$$W = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \tag{40}$$

so that $\theta = \pi/4$ and $\det W = 0$.

The mixing matrix corresponding to the maximal symmetry solution, Eq.(39), can also be written down, provided one chooses a specific phase. If we use the "standard" parametrization, then

$$V = \begin{pmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & -i\dfrac{1}{\sqrt{3}} \\ \dfrac{-e^{i\alpha}}{\sqrt{3}} & \dfrac{e^{-i\alpha}}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{e^{-i\alpha}}{\sqrt{3}} & -\dfrac{e^{i\alpha}}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \end{pmatrix}, \quad \alpha = \pi/6. \tag{41}$$

This solution is now known as being "trimaximal"[8]. It is a particular case of a bimaximal solution [9], with $\theta_{12} = \theta_{23} = \pi/4$, but $\sin\theta_{13} = 1/\sqrt{3}$ and $\delta = \pi/2$. It is noteworthy that the matrix V is complex. Indeed, this solution was known [6] to give rise to the maximally allowed value for J^2, given by Eq.(25)

$$J^2 = 1/108, \tag{42}$$

Within the present parametrization, we may demonstrate this fact by considering the variation of $J^2 = x_1 x_2 x_3 - y_1 y_2 y_3$, for arbitrary δx_i and δy_j but subject to the constraint $\sum(\delta x_i) - \sum(\delta y_j) = 0$. At a symmetric point, $x_i = -y_j$, for all (i,j), so that also $x_i x_j = y_\ell y_m$,

$$\begin{aligned} \delta J^2 &= (\delta x_1 + \delta x_2 + \delta x_3) x_i x_j - (\delta y_1 + \delta y_2 + \delta y_3) y_\ell y_m \\ &= 0. \end{aligned} \tag{43}$$

Physically, the neutrino mixing matrix is close to being bimaximal but with a small θ_{13}, and with δ completely unknown. It is tempting to speculate that there is a common origin (renormalization being a prime candidate) of the deviations of θ_{12}, θ_{13} and δ from the maximal symmetry solution. If this is correct, then, from the known physical values of θ_{12} and θ_{13}, we would have a means to calculate the phase δ.

IV.   APPLICATIONS TO V_{CKM}

For the CKM matrix, a particularly useful (approximate) parametrization is due to Wolfenstein [4], with $\lambda \simeq 0.22$,

$$
V_{CKM} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \tag{44}
$$

More accurate formulas are also available [10]. The matrix is simple in form yet it captures all of the essence of the quark mixing. Note, however, $\det V \neq 1$.

To arrive at Eq.(44), one needs to choose the phases so that only V_{13} and V_{31} are complex. We note that a rephasing invariant parametrization can be constructed in terms of the W matrix.

$$
W_{CKM} = \begin{pmatrix} 1 - a^2 - b^2 & a^2 & b^2 \\ a^2 - e^2 & 1 - a^2 - d^2 + e^2 & d^2 \\ b^2 + e^2 & d^2 - e^2 & 1 - b^2 - d^2 \end{pmatrix}. \tag{45}
$$

In this construction, we have incorporated the unitarity conditions which also implies the relations $W_{12} - W_{21} = -(W_{13} - W_{31}) = W_{23} - W_{32}$. The choice $W_{31} - W_{13} = e^2 \geq 0$ is made here in accordance with the experimental $|V_{CKM}|$ values. To make connections to V_{CKM} and to exhibit the order of magnitudes of the various parameters, we may write

$$
\begin{aligned}
a^2 &= \lambda^2, \\
b^2 &= B^2\lambda^6, \\
d^2 &= D^2\lambda^4, \\
e^2 &= E^2\lambda^6.
\end{aligned} \tag{46}
$$

These relations define $(\lambda^2, B^2, D^2, E^2)$, with (B^2, D^2, E^2) all being of order unity. We emphasize that Eq.(45), with the values given in Eq.(46), is an exact parametrization, and not an expansion in λ. Thus, Eq.(46) may be regarded as a mnemonic device to remind us of the physical values of the parameters (a^2, b^2, d^2, e^2). At the same time, once we have a quantity expressed in terms of them, it may also be used to obtain an expansion in λ. Thus, Eqs.(45,46) is a rephasing invariant generalization of the Wolfenstein parametrization, whereby higher order terms in λ can be read off directly.

From Eq.(45), we can calculate the elements of w, and hence the parameters (x_i, y_j). We have

$$
\begin{aligned}
2x_1 &= 2 - 2(b^2 + a^2 + d^2) + e^2 + \bar{w} \\
2x_2 &= e^2 + \bar{w} \\
2x_3 &= -e^2 + \bar{w} \\
2y_1 &= -2d^2 + e^2 + \bar{w} \\
2y_2 &= -2a^2 + e^2 + \bar{w} \\
2y_3 &= -2b^2 - e^2 + \bar{w}
\end{aligned}
\tag{47}
$$

where $\bar{w}$ is defined by

$$
\bar{w} = a^2 d^2 + b^2 (a^2 + d^2 - e^2).
\tag{48}
$$

Eqs.(47, 48) are exact. But it is useful to get an order of magnitude estimate by putting in the values of Eq.(46), we see that $x_1 \simeq O(1)$, $(x_2, x_3) \simeq O(\lambda^6)$, $y_1 \simeq O(\lambda^4)$, $y_2 \simeq O(\lambda^2)$, $y_3 \simeq O(\lambda^6)$. $\bar{w}$ contains terms up to $O(\lambda^{12})$, with the leading order term being $a^2 d^2$. The constraint $\sum x_i - \sum y_j = 1$ is easily verified. The constraints, Eqs.(23,24), which are valid to all orders in λ, lead to simple approximate relations (with $O(\lambda^2)$ corrections)

$$
x_1 \simeq 1 + y_1 + y_2,
\tag{49}
$$

$$
x_2 + x_3 \simeq y_1 y_2.
\tag{50}
$$

In terms of the parameterizations in Eq.(45), we can readily find the CP-violation measure

$$
J^2 = \frac{1}{4}[-a^4 d^4 - b^4 - (b^2 + e^2)^2 + 2b^2(b^2 + e^2) + 2a^2 d^2(2b^2 + e^2)] - \Delta
\tag{51}
$$

$$
\Delta = \frac{1}{4}(\bar{w} - a^2 d^2)[(\bar{w} + a^2 d^2) + 2e^2]
\tag{52}
$$

where $\bar{w}$ is defined in Eq.(48). In Eq.(51), Δ is $O(\lambda^2)$ compared to the term in the square bracket, which can be shown to be the $4 \times (area)^2$ of a triangle with sides $(ad, b, \sqrt{b^2 + e^2})$. This result is of course well-known in connection with the study of the unitarity triangles, which we will discuss in the following.

Consider the unitarity conditions, $\sum V_{ij} V_{ik}^* = \delta_{jk}$. Within the context of the present discussion, for $j = k$, they are rephasing invariant and, with Eq.(5), reduce to $\sum (\pm) \Gamma_{ijk} = 1$,

while for $j \neq k$, the conditions are rephasing dependent, but are identities if Eq.(5) is used. Thus, tests of the unitarity triangles amounts to those of Eq.(5). It is well-known that the most interesting relation is

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, \tag{53}$$

or,

$$V_{11}V_{13}^* + V_{21}V_{23}^* + V_{31}V_{33}^* = 0. \tag{54}$$

We can turn this equation into one with only rephasing invariants by multiplying, for instance, by $V_{21}^*V_{23}$:

$$V_{11}V_{13}^*V_{21}^*V_{23} + |V_{21}|^2|V_{23}|^2 + V_{31}V_{33}^*V_{21}^*V_{23} = 0 \tag{55}$$

This relation is displayed in Fig. 1. It is the usual unitarity triangle rotated and magnified (by $|V_{21}||V_{23}|$). It has a base $|V_{21}|^2|V_{23}|^2$. The other two sides are given by

$$\begin{aligned} V_{11}V_{13}^*V_{21}^*V_{23} &= -\Gamma_{312}^* + |V_{13}^2||V_{21}|^2 \\ &\cong -x_3 - iJ, \end{aligned} \tag{56}$$

$$\begin{aligned} V_{31}V_{33}^*V_{21}^*V_{23} &= -\Gamma_{231} + |V_{23}|^2|V_{31}|^2 \\ &\cong -x_2 + iJ, \end{aligned} \tag{57}$$

where we have used $|V_{13}^2|V_{21}|^2 = (x_3 - y_3)(x_3 - y_2) \ll x_3$, and $|V_{23}^2|V_{31}|^2 = (x_2 - y_1)(x_2 - y_3) \ll x_2$. Thus, the triangle in Fig. 1 has height J, with the lengths of the two sides being approximately $(x_3^2 + J^2)^{1/2}$ and $(x_2^2 + J^2)^{1/2}$. Also, the base line has length $\cong x_2 + x_3$, according to Eq.(50). It follows that

$$\tan \beta \cong J/x_2, \tag{58}$$

$$\tan \gamma \cong J/x_3. \tag{59}$$

A similar construction (by choosing a different real base line) yields
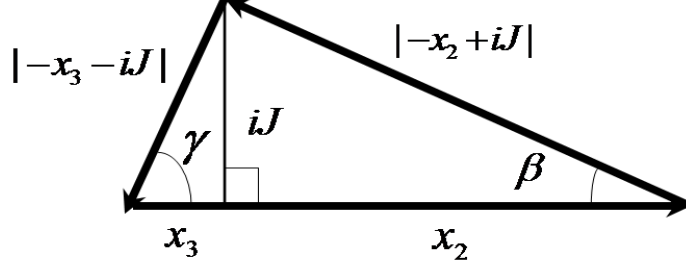
$$\tan \alpha \cong J/y_3. \tag{60}$$

FIG. 1: Magnified unitary triangle with sides $|V_{21}|^2|V_{23}|^2$, $V_{31}V_{33}^*V_{21}^*V_{23}$ and $V_{11}V_{13}^*V_{21}^*V_{23}$. Their approximate lengths are as labelled.

In other words, the angles (α, β, γ) are simply the (approximate) phase angles of the rephasing invariants $\Gamma_{231}^*, \Gamma_{312}^*$ and Γ_{321}^*, respectively.

Experimentally, CP-violating processes seem to indicate that $\alpha \cong \pi/2$ [11]. This is a very intriguing result since it implies that y_3 is much smaller than $O(\lambda^6)$, the expected "natural" value. To the extent that all of the above results are valid to $O(\lambda^2)$, we take $y_3 = O(\lambda^8)$. From Eq.(47), $\alpha \cong \pi/2$ implies that

$$a^2 d^2 \cong 2b^2 + e^2 \tag{61}$$

or,

$$|V_{us}|^2|V_{cb}|^2 \cong |V_{ub}|^2 + |V_{td}|^2 \tag{62}$$

Also, $y_3 \cong 0$ means that $x_3 \cong b^2$, $x_2 \cong b^2 + e^2$, from Eqs.(16,45). Thus, a particularly simple

set of parameters result,

$$x_1 \cong 1 - |V_{us}|^2 - |V_{cb}|^2,$$
$$x_2 \cong |V_{td}|^2,$$
$$x_3 \cong |V_{ub}|^2,$$
$$y_1 \cong -|V_{cb}|^2,$$
$$y_2 \cong -|V_{us}|^2,$$
$$y_3 \cong 0,$$

$$(63)$$

assuming $\alpha \cong \pi/2$. All above relations are accurate to $O(\lambda^2)$. In addition, for $\alpha = \pi/2$, $\tan\beta \tan\gamma = 1$, so that from Eqs.(58,59) (or from Eq.(25)) we find

$$J^2 \cong x_2 x_3 \cong |V_{td}|^2 |V_{ub}|^2. \tag{64}$$

The above relations reveal that for V_{CKM}, the parameters (x_i, y_j) have particularly simple relations with the directly measured quantities $|V_{ij}|^2$ and (α, β, γ). Whether there is a deeper meaning behind the pattern in Eq.(63) remains to be seen.

V. CONCLUSION

In this paper, we propose to parameterize a three flavor mixing matrix by Γ_{ijk} (Eq. (2)), which are rephasing invariant when we demand, without loss of generality, that $\det V = 1$. All of the Γ's have the same imaginary part, $-J$, which is the CP-violation measure. The six real parts of Γ satisfy two constraints (Eqs.(23, 24)), resulting in four independent ones, as expected. In addition, J^2 is given in a very symmetric expression, Eq. (25).

The Γ-parametrization is characterized by its symmetry, which is a reflection of the inherent property of the three-flavor mixing. With its help we are able to identify a mixing pattern of "maximal symmetry", in Eqs.(39, 41). Its resemblance to the neutrino mixing matrix seems to suggest a possible origin of the latter. This possibility will be explored.

The relation between the (x_i, y_j) parameters and $|V_{ij}|^2$ was discussed in detail. As an application we find explicit (x_i, y_j) values corresponding to the physical V_{CKM}. It is shown that all of the measurable quantities $(|V_{ij}|^2$, phase angle$(\alpha, \beta, \gamma))$ are directly related to the (x_i, y_j) variables. Of the three x-values, one is close to unity and the other two are small $(O(\lambda^6))$, while the three y-values are of order $O(\lambda^2), O(\lambda^4)$ and $O(\lambda^8)$, respectively. To a good approximation, $\alpha \cong \pi/2$, it is found that (x_2, x_3, y_1, y_2) are simply equal to $(|V_{td}|^2, |V_{ub}|^2, -|V_{cb}|^2, -|V_{us}|^2)$.

The use of rephasing invariants should be useful in other problems, for instance, in parameterizing the mass matrices. We hope to return to this topic in the future.

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